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TURBULENCE IN STARS. II. SHEAR, STABLE STRATIFICATION, AND RADIATIVE LOSSES

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ABSTRACT

For many years shear-driven turbulence was thought to provide sufficient turbulent mixing in stably stratified regions to explain stellar structure data. It has recently been argued that the mixing is too weak and that alternative mixing mechanisms are required. The conclusion is predicated, among others, on the key assumption that turbulence exists only for $\mathrm{Ri} < \mathrm{Ri}_{\mathrm{cr}} = \frac{1}{4}$. This result follows from linear stability analysis and contradicts a variety of data. We suggest a new definition of $\mathrm{Ri}_{\mathrm{cr}}$: *it is the value of Ri at which the turbulent kinetic energy vanishes*. We find that for $\mathrm{Pe} > 1$ (no radiative losses) $\mathrm{Ri}_{\mathrm{cr}} \sim 1$, while for $\mathrm{Pe} < 1$ (important radiative losses) $\mathrm{Ri}_{\mathrm{cr}} \sim \mathrm{Pe}^{-1} \geq 1$. Thus, we find more mixing. We present an internally consistent treatment of all the physical variables, individually and collectively, and show that turbulence is alive and well above the $\mathrm{Ri} > \frac{1}{4}$ limit. However, without a specific application of the model to a stellar case, we cannot claim that the new model will provide the mixing required by stellar data.

Subject headings: hydrodynamics — stars: interiors — turbulence

1. INTRODUCTION

It has long been thought that shear-driven turbulence could provide the turbulent mixing necessary to explain a variety of stellar data (Schatzman 1969; Zahn 1992, 1993, 1994). Recently, however, several authors have voiced dissatisfaction about the resulting mixing being too weak (Schatzman & Baglin 1991; Maeder 1995, 1996, 1997; Maeder & Meynet 1996; Pinsonneault et al. 1990a, 1990b; Pinsonneault 1997; Chaboyer, Demarque, & Pinsonneault 1995; Talon et al. 1997). We reanalyze the problem using a turbulence model which, in contrast with phenomenological models, does not require patch-up work and, perhaps most importantly, has a proven record of performance.

The gist of the results is as follows: the upper limit of the Richardson number $\mathrm{Ri}_{\mathrm{cr}}$ above which turbulence no longer exists is not given by $\mathrm{Ri}_{\mathrm{cr}} = \frac{1}{4}$ as assumed thus far. We suggest defining $\mathrm{Ri}_{\mathrm{cr}}$ as the value at which the turbulent kinetic energy vanishes. It follows that when $\mathrm{Pe} > 1$ (negligible radiative losses), $\mathrm{Ri}_{\mathrm{cr}} \sim 1$, in agreement with the data; when $\mathrm{Pe} < 1$ (important radiative losses), $\mathrm{Ri}_{\mathrm{cr}} \sim \mathrm{Pe}^{-1} > 1$. In both regimes, turbulence exists beyond the $\mathrm{Ri} = \frac{1}{4}$ limit. This provides more mixing.

Before we present the new model, it is important to gain some physical insight into the problem. Consider the Reynolds stress $\overline{uw}$ which is related to the mean shear by a turbulent momentum diffusivity ν_T:

$$\overline{uw} = -\nu_T \frac{\partial U}{\partial z} . \tag{1a}$$

Since ν_T has dimensions of velocity times length, it is customary to write it as

$$\nu_T = C_\nu \frac{K^2}{\epsilon} , \tag{1b}$$

where

$$C_\nu = C_\nu(\mathrm{Ri}, \mathrm{Pe}) . \tag{1c}$$

Here the Richardson number Ri characterizes the degree of stratification and the Peclet number $\mathrm{Pe} = wl\chi^{-1}$ characterizes the importance of radiative losses; K is the turbulent kinetic energy and ϵ is the rate of dissipation of K. Since K and ϵ are solutions of two well-known differential equations (see eqs. [48]–[49]), this is known as the K-ϵ model widely used to study shear-driven turbulence (for an extensive list of applications, see Rodi 1984). We must also include the heat flux (in units of $c_p \rho$)

$$\overline{w\theta} = \chi_T \beta , \tag{2a}$$

where the turbulent heat diffusivity χ_T is written as

$$\chi_T = C_h \frac{K^2}{\epsilon} \tag{2b}$$

and

$$C_h = C_h(\mathrm{Ri}, \mathrm{Pe}) . \tag{2c}$$

The turbulent Prandtl number σ_T is defined as the ratio

$$\sigma_T = \frac{\nu_T}{\chi_T} = \frac{C_\nu}{C_h} , \tag{2d}$$

where

$$\sigma_T = \sigma_T(\mathrm{Ri}, \mathrm{Pe}) .$$

The Richardson number Ri (e.g., Maslowe 1981; Tritton & Davies 1981) is defined as the ratio of the squares of the Brunt-Väisälä frequency N to the mean shear Σ,

$$\mathrm{Ri} \equiv \frac{N^2}{\Sigma^2} , \tag{2e}$$

where

$$N^2 \equiv -g\alpha\beta ,$$

$$\beta \equiv -\frac{\partial T}{\partial z} + \left(\frac{\partial T}{\partial z}\right)_{\mathrm{ad}} = TH_p^{-1}(\nabla - \nabla_{\mathrm{ad}}) , \tag{2f}$$

and α is the thermal expansion coefficient. The mean shear Σ is defined as

$$\Sigma = (2S_{ij}S_{ij})^{1/2} , \quad S_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right) , \tag{2g}$$

where S_{ij} is the shear and U_i is the large-scale velocity field.

What is the expected form of $C_v(\mathrm{Ri}, \mathrm{Pe})$ and $C_h(\mathrm{Ri}, \mathrm{Pe})$? First, they must satisfy the well-known asymptotic limits of $\mathrm{Ri} \to 0$ (pure shear) and $\mathrm{Ri} \to \infty$ (convection); in the first case, the model must reproduce the empirical value

$$C_v(0, \infty) \approx 0.1 , \tag{2h}$$

and in the second case it must reproduce the thermal convection results (§ 16):

$$\mathrm{Pe} > 1: \quad \frac{\chi_T}{\chi} \sim \mathrm{Pe} ,$$

$$\mathrm{Pe} < 1: \quad \frac{\chi_T}{\chi} \sim \mathrm{Pe}^2 . \tag{2i}$$

Let us now consider the overall structure of the functions $C_{v,h}$. In the case of ocean turbulence, stratification effects are unimpeded since there are no radiative losses $\mathrm{Pe} > 1$, and C_v is expected to decrease with Ri. The critical question is to determine the value $\mathrm{Ri}_{\mathrm{cr}}$ at which C_v vanishes. Ocean data suggest $\mathrm{Ri}_{\mathrm{cr}} \geq 1$. When radiative losses are important, $\mathrm{Pe} < 1$, stratification loses its power to damp turbulence, and for a given Ri, C_v should increase as Pe decreases; thus, the value of $\mathrm{Ri}_{\mathrm{cr}}$ is larger than values of Ri than in the $\mathrm{Pe} > 1$ case. The precise value of $\mathrm{Ri}_{\mathrm{cr}}$ can be derived only from solving the full turbulence problem. Model results are presented in Figure 1.

Next, consider $C_h(\mathrm{Ri}, \mathrm{Pe})$. Here we expect a behavior quite different from that of $C_v(\mathrm{Ri}, \mathrm{Pe})$. In fact, at any given Ri, the increase in radiative losses (decreases of Pe) weakens the correlations among velocity and temperature fields and thus C_h must decrease correspondingly, just the opposite of what happens to C_v. If this behavior persists all the way to $\mathrm{Ri} \to 0$, then the curves corresponding to different Pe cannot begin at the same point as in the C_v case. This would imply that at $\mathrm{Ri} \to 0$,

$$\mathrm{Pe} > 1: \quad C_h(0, \mathrm{Pe}) \sim \mathrm{Pe}^0 ,$$

$$\mathrm{Pe} < 1: \quad C_h(0, \mathrm{Pe}) \sim \mathrm{Pe} . \tag{2j}$$

The model results are presented in Figure 2 (see also § 12).

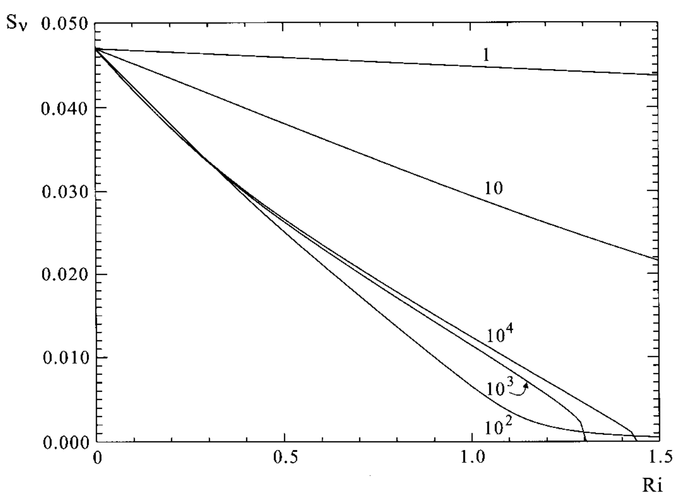

FIG. 1.—Model predictions for the function $C_v(\mathrm{Ri}, \mathrm{Pe}) = 2S_v$; see eqs. (1b) and (44a). The Peclet number Pe is defined in eqs. (15), (53a) and (53b). Pe_0 is treated as a free parameter to highlight the sensitivity of C_v to radiative losses. As explained in the text, oceanic data ($\mathrm{Pe} > 1$) show that turbulence exits up to $\mathrm{Ri} \sim 1$, as predicted by this model. As radiative losses become important ($\mathrm{Pe} < 1$), the effect of stratification that weakens turbulence decreases, and one has more mixing, represented by a larger C_v.

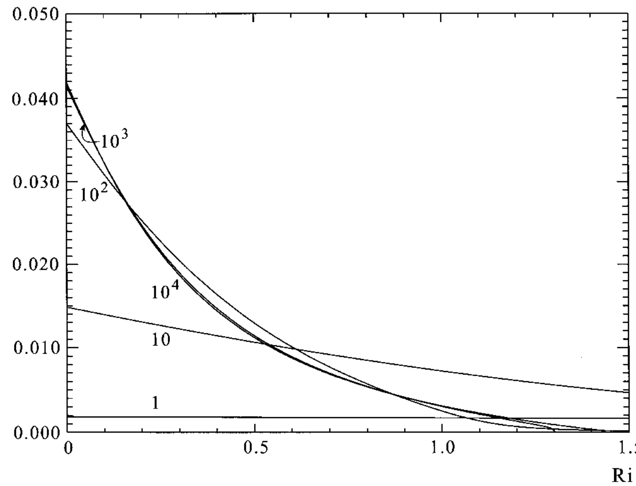

FIG. 2.—Same as in Fig. 1, but for the function $C_h(\mathrm{Ri}, \mathrm{Pe}) = 2S_h$; see eqs. (2b) and (44a). Contrary to the case of momentum diffusivity, an increase in the effect of radiative losses decreases C_h.

Concerning $\mathrm{Ri}_{\mathrm{cr}}$, we analyze two well-known suggestions and then suggest a new criterion. First, consider the standard assumption that turbulence exists only for

$$\mathrm{Ri} < \mathrm{Ri}_{\mathrm{cr}} , \quad \mathrm{Ri}_{\mathrm{cr}} = \tfrac{1}{4} . \tag{3a}$$

A problem with equation (3a) arose when atmospheric turbulence was found to exist for $\mathrm{Ri} > \frac{1}{4}$. Townsend (1958a, 1958b) invoked the effect of radiative losses; that is the case $\mathrm{Pe} < 1$. He suggested that in that case there is a renormalization of Ri, to

$$\mathrm{Ri} \to \mathrm{Ri}_{\mathrm{eff}} \equiv \mathrm{Ri}\,\mathrm{Pe} \tag{3b}$$

and that equation (3a) changes to

$$\mathrm{Ri}\,\mathrm{Pe} < \mathrm{Ri}_{\mathrm{cr}} , \quad \mathrm{Ri}_{\mathrm{cr}} = \tfrac{1}{4} . \tag{3c}$$

Since $\mathrm{Pe} < 1$, even if $\mathrm{Ri} > \frac{1}{4}$, equation (3c) is satisfied. At first sight, the renormalization equation (3b) has an intuitive appeal since radiative losses erode the temperature gradient and thus weaken its role as a sink of turbulence. There is, however, an observational fact that cannot be explained by radiative losses alone: in the ocean there are no radiative losses, and yet equation (3a) is violated: in fact, turbulence exists up to $\mathrm{Ri} \sim 1$. The earliest laboratory data seem to be by Taylor (see Monin & Yaglom 1971) who found that considerable turbulent exchange may exist even when $\mathrm{Ri} \sim 10$. More recently, Martin (1985) and Smart (1988) have found considerable mixing in the ocean up to $\mathrm{Ri} \sim 1$, a result also validated by the LES results of Wang, Large, & McWilliams (1997). This means that, independently of the effects of radiative losses, a model for stably stratified turbulence must be able to explain the existence of turbulence past the $\mathrm{Ri}_{\mathrm{cr}} = \frac{1}{4}$ limit. As shown in Figure 1, the new model predicts $\mathrm{Ri}_{\mathrm{cr}} \geq 1$, in agreement with ocean data.

Next, consider the flux Richardson number R_f (Townsend 1958a, 1958b). In the stationary and no-diffusion limit, production P of turbulence must equal its rate of dissipation ϵ. Since $P = P_s + P_b$, where

$$P_s = -\overline{uw}\,\frac{\partial U}{\partial z} , \quad P_b = g\alpha\overline{w\theta} , \tag{3d}$$

the $P = \epsilon$ condition, using equations (1a) and (2a) becomes

$$\nu_T \Sigma^2 (1 - \mathrm{R}_f) = \epsilon , \tag{3e}$$

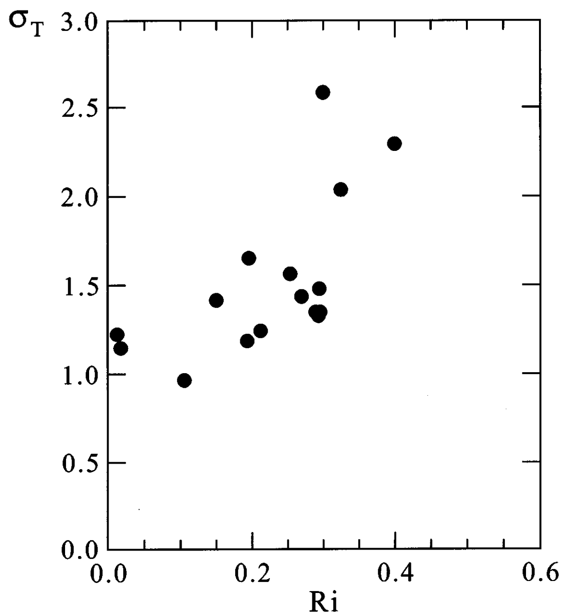

FIG. 3.—Turbulent Prandtl number σ_T is defined as the ratio of the momentum to the temperature turbulent diffusivities and is given by the ratio C_v/C_h. For the case of no radiative losses, we reproduce the laboratory data of Webster (1964).

where the flux Richardson number R_f is defined as

$$R_f = \frac{\text{Ri}}{\sigma_T} . \tag{3f}$$

On the basis of equation (3e), Townsend (1958a, 1958b) suggested the condition

$$R_f \leq 1 . \tag{3g}$$

We now show that equation (3g) is naturally satisfied. Specifically, we show that

$$\text{Ri} \to \infty: \quad R_f \to \text{constant} < 1 . \tag{3h}$$

To prove equation (3h) without carrying out a full computation, we recall that when radiative losses are unimportant, laboratory data (Webster 1964; Istweire & Helland 1989) and numerical simulations (Schumann & Gerz 1995; Wang et al. 1996) show that stable stratification reduces the transfer of heat more than that of momentum, (see Fig. 3). Even the simple fit

$$\sigma_T(\text{Ri}, \infty) = \sigma_T(0) + a\text{Ri} \tag{3i}$$

with $a > 1$ gives

$$\text{Ri} \to \infty: \quad R_f \to a^{-1} < 1 , \tag{3j}$$

in agreement with equation (3h).

To quantify the dependence of σ_T on Pe, we use the $\text{Ri} \to \infty$ data corresponding to the case of convection and equation (2a) rewritten as

$$\overline{w\theta} = \chi_T(\text{Pe}) \, |\beta| . \tag{4a}$$

From RNG (renormalization group; Canuto & Dubovikov 1998), we know that

$$\text{Pe} > 1: \quad \frac{\chi_T}{\chi} \sim \text{Pe} ,$$

$$\text{Pe} < 1: \quad \frac{\chi_T}{\chi} \sim \text{Pe}^2 , \tag{4b}$$

If we assume that ν_T does not depend on χ since to first order, momentum transfer is immune from radiative process, equation (4b) suggests that

$$\text{Pe} < 1: \quad \sigma_T(\text{Ri}, \text{Pe}) \sim \sigma_T(\text{Ri})\text{Pe}^{-1} . \tag{4c}$$

Thus,

$$\text{Ri} \to \infty: \quad R_f \to a^{-1}\text{Pe} < 1 ,$$

which also satisfies equation (3h). The laboratory data are presented in Figure 4 (Linden 1979, 1980; Maderich, Konovalov, & Konstantinov 1995).

We propose that the true Ri_{cr} is set by the behavior of the turbulent kinetic energy K. When radiative losses are important, stratification is much less effective and K decreases with Ri much more slowly. K vanishes at a value of Ri above which turbulence no longer exists. Thus, we shall define Ri_{cr} as the solution of the equation

$$K(\text{Ri}_{\text{cr}}) = 0 , \tag{5a}$$

with

$$\text{Ri}_{\text{cr}} = \text{Ri}_{\text{cr}}(\text{Pe}) . \tag{5b}$$

We expect the following conditions to be satisfied:

$$\text{Ri}_{\text{cr}}(\text{Pe} < 1) > \text{Ri}_{\text{cr}}(\infty) > \tfrac{1}{4} . \tag{5c}$$

The last condition is demanded by the ocean data discussed earlier, while the first inequality is required by physical considerations.

Finally, we discuss the temperature gradient. None of the above turbulence variables is a measurable quantity as such, only the final temperature gradient is, for example, with helioseismological data that provide the sound speed (Canuto & Christensen-Dalsgaard 1998). The standard flux conservation law (F_r is the radiative flux),

$$F_r + F_C = \text{const} , \tag{6a}$$

is no longer valid since Reynolds stresses can be transported by the large-scale flow giving rise to a flux $U_j \tau_{ij}$ which must enter the flux conservation law. In fact, the dynamic equation for the mean temperature (see eq. [18]) below, depends on U_i. As a consequence, the new flux conservation law

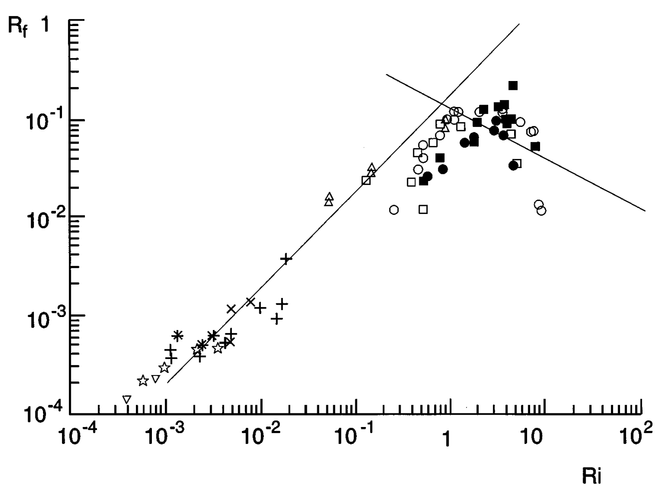

FIG. 4.—Flux Richardson number R_f vs. Ri as from a variety of laboratory data (redrawn from Maderich et al. 1995). The symbols refer to different authors cited in Maderich et al. (1995) and refer to either grid generated turbulence and/or freely decaying turbulence in a stratified medium.

reads

$$F_i^r + F_i^c + F_i^{\kappa e} + \rho U_j[(c_p T + K_u + K + G)\delta_{ij} + \tau_{ij}] = \text{constant} . \quad (6b)$$

What is conserved is the sum of the radiative, convective, and turbulent kinetic energy ($F^{\kappa e}$) fluxes *plus* a new term that depends on the large-scale flow, U_i; $K_u = \frac{1}{2}U_i^2$ is the kinetic energy of the large-scale flow, G is the gravitational field, and τ_{ij} are the Reynolds stresses representing turbulence. As an example, suppose $i = 3$, $\boldsymbol{U} = U(z), V(z), 0$, and use equations (1a), (2a), and (2f). Using

$$F_r = K_r T H_p^{-1} \nabla , \quad F_c = c_p \rho \overline{w\theta} , \quad K_r = c_p \rho \chi , \quad (7a)$$

we obtain

$$\chi \nabla + \chi_T(\nabla - \nabla_{\rm ad}) = \chi \nabla_r^* , \quad (7b)$$

where the new radiative ∇_r^* is given by

$$\nabla_r^* \equiv \nabla_r + H_p(K_r T)^{-1}\left(\rho \nu_T \frac{\partial}{\partial z} K_u - F^{\kappa e}\right) . \quad (7c)$$

The turbulence model presented here satisfies all the above requirements. It was constructed using renormalization group methods (RNG) and new data from laboratory, direct and large eddy simulations (DNS and LES, respectively) of shear, convection, stable stratification, etc. In this new model: (1) all the dynamic equations governing the turbulent variables are derived from the Navier-Stokes equations and the entropy equations, (2) the inclusion of shear, buoyancy rotation, etc., does not require additional assumptions since the general rules are set at the beginning, (3) the closures required to treat the higher order moments have been tested on different flows, (4) the extension to cases of astrophysical interest (low Prandtl number, small Peclet number, etc.) is carried out by employing a two-point closure model that allows a proper calculation of the relevant timescales that depend on the Peclet number. We present four turbulence models: (1) all the turbulent variables satisfy dynamic equations, (2) only K and its rate of dissipation ϵ satisfy dynamic equations while all the other turbulent variables are given in algebraic form, (3) only ϵ satisfies a dynamic equation while all the other turbulent variables are algebraic, and (4) all the turbulent variables are given in algebraic form.

2. MAIN PHYSICAL PARAMETERS

In order to describe the flow under consideration, we need to introduce several variables. Much as the large-scale variables are characterized by N and Σ, shear-driven, stratified turbulence is characterized by a *shear number*,

$$\mathrm{Sh} = l\Sigma K^{-1/2} = \frac{K}{\epsilon}\Sigma , \quad (8)$$

and a *Froude number*,

$$\mathrm{Fr} = \frac{K^{1/2}}{lN} = \frac{\epsilon}{K} N^{-1} . \quad (9)$$

The *Richardson number* Ri is defined as

$$\mathrm{Ri} \equiv (\mathrm{FrSh})^{-2} = \frac{N^2}{\Sigma^2} . \quad (10)$$

Fr quantities the role of nonlinear interactions versus the effect of stratification,

$$\mathrm{Fr} = \frac{v^2 l^{-1}}{Nv} = \frac{v}{lN} = \frac{K^{1/2}}{l} N^{-1} \quad (11)$$

in the same way that the Reynolds number quantifies the role of nonlinear interactions versus viscous forces,

$$\mathrm{Re} = \frac{v^2 l^{-1}}{\nu v l^{-2}} = \frac{vl}{\nu} = \frac{K^{1/2} l}{\nu} . \quad (12)$$

When $\mathrm{Fr} > 1$, turbulent flows are only slightly affected by stratification, and the inertial Kolmogorov spectrum applies; when $\mathrm{Fr} < 1$, turbulence is affected by stratification, and $\mathrm{Fr} = 1$ is the demarcation between the two regimes (often called "turbulence" and "wave" regimes). $\mathrm{Fr} = 1$ occurs at the Ozmidov length scale (Dougherty 1961; Lumley 1964; Ozmidov 1965; Hunt, Kaimal, & Gaynor 1985)

$$l_0 = \frac{K^{1/2}}{N} = \left(\frac{\epsilon}{N^3}\right)^{1/2} . \quad (13)$$

Scales $l \geq l_0$ are strongly affected by stratification, while those with $l \leq l_0$ are not. The problem also possesses four timescales:

$$\frac{1}{2}\tau = \frac{K}{\epsilon} , \quad \tau_\theta = \frac{\bar{\theta}^2}{\epsilon_\theta} , \quad \tau_N = N^{-1} , \quad \tau_\Sigma = \Sigma^{-1} . \quad (14)$$

Finally, the Peclet number is defined as the ratio of the radiative timescale $l^2\chi^{-1}$ to the turbulence timescale τ:

$$\mathrm{Pe} = \frac{1}{2}\frac{\epsilon l^2}{\chi K} = \frac{1}{2} c_\epsilon^2 \frac{K^2}{\epsilon\chi} = \frac{4\pi^2}{125}\frac{K^2}{\epsilon\chi} . \quad (15)$$

Here $K = (3\mathrm{Ko}/2)\epsilon^{2/3}k_0^{-2/3}$, $l = \pi/k_0$, $\mathrm{Ko} = 5/3$ is the Kolmogorov constant, and

$$c_\epsilon = \pi\left(\frac{2}{3\mathrm{Ko}}\right)^{3/2} . \quad (16)$$

3. LARGE-SCALE DYNAMICS

The large-scale flow is characterized by a velocity field U_i which satisfies the following dynamic equation ($D/Dt \equiv \partial/\partial t + U_j \partial/\partial x_j$):

$$\frac{D}{Dt} U_i = -g_i - \frac{\partial}{\partial x_j}(\delta_{ij} P + \tau_{ij}) . \quad (17)$$

The Reynolds stresses are seen to contribute to the gas pressure.

The second equation is for the mean temperature T which reads (Canuto 1997)

$$\rho \frac{D}{Dt}(c_p T + K_u + K + G) = -\frac{\partial}{\partial x_i}(F_i^c + F_i^r + F_i^{\kappa e} + \rho\tau_{ij} U_j) + \frac{\partial}{\partial t} P . \quad (18)$$

Here $K = \frac{1}{2}\tau_{ii}$ and $K_u \equiv \frac{1}{2}U_i^2$ are the kinetic energies (per unit mass) of turbulence and of the large-scale flow; G is the gravitational field,

$$g_i U_i \equiv \frac{DG}{Dt} , \quad (19)$$

and F_i^r is the radiative flux. The convective and turbulent kinetic energy fluxes are defined as

$$F_i^c \equiv c_p \rho \overline{u_i \theta} , \quad F_i^{\kappa e} = \tfrac{1}{2}\rho q^2 u_i , \quad \tfrac{1}{2} q^2 \equiv u_i u_i . \tag{20}$$

In the stationary case, $\partial/\partial t = 0$, we obtain equation (6b).

As one can see, the equations for the large-scale fields U_i and T depend on turbulence via the Reynolds stresses τ_{ij} and the convective fluxes F_i^c which must be provided by a turbulence model.

4. TURBULENCE MODEL

Turbulence models can be divided into two categories: one-point closures and two-point closures. The first type has been widely employed in the study of engineering flows, while the second has been developed mostly to study tubulence as such but rarely has the model been used in practical applications. The gap has been closed recently by the work of Canuto & Dubovikov (1996a, 1996b, 1996c, 1997a, 1997b, 1997c, 1998) who used the renormalization group to derive a two-point closure model which is then integrated over all wavenumbers to obtain the one-point closure which can be directly used in the astrophysical context. The difficulties of closure and the state of the art models are discussed in Canuto (1992, 1993, 1994).

In addition to the large-scale components (T, U_i), the velocity and temperature fields contain fluctuating parts (θ, u_i), with $\bar{\theta} = \bar{u}_i = 0$. With them, one constructs the following physical variables (second-order moments).

$$\overline{u_i u_j} = \tau_{ij} \qquad \text{(Reynolds stresses)} , \tag{21}$$

$$K \equiv \tfrac{1}{2}\tau_{ii} \qquad \text{(turbulent kinetic energy)},$$

$$\overline{u_i \theta} = h_i \qquad \text{(convective fluxes)} , \tag{22}$$

$$\bar{\theta}^2 = \qquad \text{(temperature variance)} , \tag{23}$$

$$\epsilon = \nu \overline{\left(\frac{\partial u_i}{\partial x_j}\right)^2} , \quad \epsilon_\theta = \chi \overline{\left(\frac{\partial \theta}{\partial x_i}\right)^2}$$

$$\text{(dissipation rates of } K \text{ and } \bar{\theta}^2) , \tag{24}$$

for a total of $6 + 3 + 1 + 2 = 12$ variables, each of which satisfies a separate dynamic equation, and each of which depends on the others so that it is not possible to construct an expression for the convective flux without solving the equation for the Reynolds stresses, and vice versa.

1. Turbulent kinetic energy, K:

$$\frac{DK}{Dt} + D_f(K) = P_s + P_b - \epsilon , \tag{25a}$$

where $\lambda_i \equiv g_i \alpha$,

$$P_s \equiv -\tau_{ij} S_{ij} , \quad P_b \equiv \lambda_i h_i , \tag{25b}$$

$$D_f(K) = \frac{\partial}{\partial x_i} F_i^{\kappa e} , \tag{25c}$$

P_s and P_b are the rates of production of K by shear and buoyancy, and $D_f(K)$ is the transport of the flux of turbulent kinetic energy (per unit mass) defined in equation (20).

2. Temperature variance, $\overline{\theta^2}$:

$$\frac{D}{Dt}\frac{1}{2}\bar{\theta}^2 + D_f\left(\frac{1}{2}\bar{\theta}^2\right) = P_\theta - \epsilon_\theta + \frac{\partial}{\partial x_i}\left(\chi \frac{\partial}{\partial x_i}\frac{1}{2}\bar{\theta}^2\right) , \tag{26a}$$

$$P_\theta = h_i \beta_i , \quad D_f\left(\frac{1}{2}\bar{\theta}^2\right) = \frac{\partial}{\partial x_i}\left(\frac{1}{2}\overline{u_i \theta^2}\right) , \tag{26b}$$

and P_θ and D_f are the rates of production and transport of the flux of temperature variance.

3. Reynolds stresses, $b_{ij} \equiv \tau_{ij} - \frac{2}{3}K\delta_{ij}$:

$$\frac{D}{D_t} b_{ij} + D_f(b_{ij}) = -\frac{8}{15} KS_{ij} - 2\tau_{pv}^{-1} b_{ij} + \beta_5 B_{ij}$$

$$- (1 - \alpha_1)\Sigma_{ij} - (1 - \alpha_2)Z_{ij} . \tag{27a}$$

The traceless tensors $\boldsymbol{B}$, $\boldsymbol{\Sigma}$, and $\boldsymbol{Z}$ are defined as

$$B_{ij} = \lambda_i h_j + \lambda_j h_i - \tfrac{2}{3}\delta_{ij}\lambda_k h_k , \tag{27b}$$

$$\Sigma_{ij} = S_{ik} b_{kj} + S_{jk} b_{ik} - \tfrac{2}{3}\delta_{ij} S_{kl} b_{kl} , \tag{27c}$$

$$Z_{ij} = V_{ik} b_{kj} + V_{jk} b_{ik} . \tag{27d}$$

The shear S_{ij} is defined in equation (2g) while the mean vorticity tensor V_{ij} is defined by

$$V_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} - \frac{\partial U_j}{\partial x_i}\right) . \tag{27e}$$

The diffusion term is defined as

$$F_f(b_{ij}) = \frac{\partial}{\partial x_k}\left(\overline{u_i u_j u_k} \frac{1}{3}\delta_{ij}\overline{q^2 u_k}\right) . \tag{27f}$$

4. Heat flux, $h_i = \overline{u_i \theta}$:

$$\frac{D}{Dt} h_i + D_f(h_i) = \tau_{ij}\beta_j - \left(1 - \frac{3}{4}\alpha_3\right) h_j S_{ij}$$

$$- \left(1 - \frac{5}{4}\alpha_3\right) h_j V_{ij} + (1 - \gamma_1)\lambda_1 \bar{\theta}^2$$

$$- \tau_{p\theta}^{-1} h_i + \frac{\partial}{\partial x_j}\left(\chi \frac{\partial}{\partial x_j} h_i\right) , \tag{28a}$$

with

$$D_f(h_i) = \frac{\partial}{\partial x_j}(\overline{u_i u_j \theta} + \delta_{ij}\overline{p\theta}) . \tag{28b}$$

5. Dissipation rate, ϵ_θ:

$$\epsilon_\theta = \tau_\theta^{-1}\bar{\theta}^2 . \tag{29}$$

6. Dissipation rate, ϵ:

$$\frac{D\epsilon}{Dt} + D_f(\epsilon) = \epsilon K^{-1}(c_1 P_s + c_3 P_b) - c_2 \epsilon^2 K^{-1} , \tag{30a}$$

$$D_f(\epsilon) = \frac{\partial}{\partial x_i}(\overline{\epsilon u_i}) . \tag{30b}$$

Since we have assumed an algebraic for ϵ_θ, the number of dynamic equations is 11.

5. CRITICAL INGREDIENT: TIMESCALES AND THEIR PECLET NUMBER DEPENDENCE

It is clear that the above equations cannot be solved unless the turbulence model provides the timescales τ_{pv}, $\tau_{p\theta}$, and τ_θ. They originate in the following way (Canuto 1992, 1993, 1994). The Navier-Stokes equations for the turbulent velocity field contain pressure terms $\partial p/\partial x_i \equiv p_{,i}$ which, when one generates the equations for b_{ij} and h_i, bring into the problem vectors and tensors of the type

$$\Pi_i \equiv \overline{\theta p_{,i}}\,, \quad \Pi_{ij} \equiv \overline{u_j p_{,i}} + \overline{u_j p_{,i}}\,. \tag{31}$$

Since the fluctuating pressure p does not satisfy the hydrostatic equilibrium equation, these higher order terms must be modeled in terms of second-order terms. The process brings about the timescales $\tau_{p\theta}$ (pressure-temperature) and τ_{pv} (pressure-velocity), as well as the constants α's and γ_1. The timescale τ_θ was defined in equation (29). Since K and ϵ satisfy equations (25a) and (30), the dynamic timescale $\tau = 2K/\epsilon$ is known. The question then arises as to what are the relations

$$(\tau_\theta, \tau_{p\theta}, \tau_{pv}) \text{ versus } \tau\,. \tag{32}$$

It is quite clear that since the radiative timescale may become shorter than τ, equation (32) cannot be satisfied with only constants of proportionality. One must go beyond the present one-point formalism and employ a two-point closure (Canuto & Dubovikov 1998). The results are as follows:

$$\frac{\tau_{pv}}{\tau} = \frac{2}{5}\,, \tag{33a}$$

$$\frac{\tau_{p\theta}}{\tau} = (4\pi^2)^{-1}\,\mathrm{Pe}\,[1 + 5(4\pi^2)^{-1}(1 + \sigma_t^{-1})\,\mathrm{Pe}]^{-1}\,, \tag{33b}$$

$$\frac{\tau_\theta}{\tau} = 4(7\pi^2)^{-1}\,\mathrm{Pe}\,[1 + 4(7\pi^2)^{-1}\sigma_t^{-1}\,\mathrm{Pe}]^{-1}\,. \tag{33c}$$

The turbulent Prandtl number $\sigma_t = \sigma_t(\mathrm{Pe})$ as a function of Pe is given by

$$b\sigma_t^{-1} = 1 + \frac{2}{5}\pi^2\,\mathrm{Pe}^{-1}\,(b - \sigma) \times \left[\left(1 + \frac{5}{2}\pi^{-2}\,\mathrm{Pe}\,\frac{a\sigma_t^{-1} + 1}{a + \sigma}\right)^{-a/b} - 1\right], \tag{34a}$$

where σ is the Prandtl number, typically $\sigma \ll 1$. The constants a and b are given by

$$2a = (\gamma^2 + 4\gamma)^{1/2} - \gamma\,, \quad b = a + \gamma\,. \tag{34b}$$

With $\gamma = 3/10$, we have

$$\frac{a}{b} = 0.58\,, \quad b = 0.72\,. \tag{34c}$$

It is clear from equations (33)–(34a) that

$$\mathrm{Pe} > 1: \quad \left(\frac{\tau_{pv}}{\tau}, \frac{\tau_{p\theta}}{\tau}, \frac{\tau_\theta}{\tau}\right) \sim \mathrm{Pe}^0\,, \tag{35a}$$

whereas for

$$\mathrm{Pe} < 1: \quad \frac{\tau_{pv}}{\tau} \sim \mathrm{Pe}^0\,, \quad \left(\frac{\tau_{p\theta}}{\tau}, \frac{\tau_\theta}{\tau}\right) \sim \mathrm{Pe}\,. \tag{35b}$$

Equation (35b) confirms what we assumed in deriving equation (4c), radiative losses affect the temperature field more than the momentum field. We recall that Pe is defined by equation (15).

6. DIFFUSION TERMS

The third-order moments, equations (25c), (26b), (27f), and (28b), satisfy the dynamical equations derived earlier (Canuto 1992, eqs. [55a]–[55d]; Appendix A).

7. ALGEBRAIC MODELS

Although the model presented thus far is complete and allows the calculation of the turbulent variables of interest, it is obviously rather complex since it entails 11 different equations. In this section, we present a model in which we retain only two of the 11 differential equations, those for K and ϵ. In the remaining equations, we neglect the D/Dt term and the diffusion terms. After some algebra we obtain

Reynolds stresses:

$$b_{ij} = -\tfrac{4}{15}K\tau_{pv}S_{ij} + \tfrac{1}{2}\beta_5\,\tau_{pv}B_{ij} - \tfrac{1}{2}\tau_{pv}(1 - \alpha_1)\Sigma_{ij} - \tfrac{1}{2}\tau_{pv}(1 - \alpha_2)Z_{ij}\,. \tag{36}$$

Convective fluxes:

$$A_{ik}h_k = (\chi_T)_{ij}\beta_j\,. \tag{37a}$$

The turbulent conductivity tensor is given by

$$(\chi_T)_{ij} = \tau(b_{ij} + \tfrac{2}{3}\delta_{ij}K) \tag{37b}$$

and

$$A_{ij} = \frac{\tau}{\tau_{p\theta}}\,\delta_{ij} - (1 - \gamma_1)\tau\tau_\theta\,\lambda_i\beta_j + \left(1 - \frac{3}{4}\,\alpha_3\right)\tau S_{ij} + \left(1 - \frac{5}{4}\,\alpha_3\right)\tau V_{ij}\,. \tag{37c}$$

The fact that the right-hand side of equation (37b) depends on b_{ij} itself makes the analytic solution in the three-dimensional case somewhat cumbersome although manageable with symbolic algebra.

We recall that the "standard" model is an approximation to equations (36) and (37) that retains only the first terms in equation (36):

$$b_{ij} = -\nu_T S_{ij}\,, \quad \nu_T = \frac{16}{75}\frac{K^2}{\epsilon}\,, \tag{37d}$$

while on the right-hand side of equation (37b) one retains only the isotropic term, and the matrix **A** is taken to be diagonal. Thus,

$$h_i = \chi_T\beta_i\,, \quad \chi_T = \frac{4}{3}\left(\frac{\tau_{p\theta}}{\tau}\right)\frac{K^2}{\epsilon}\,. \tag{37e}$$

In Canuto & Christensen–Dalsgaard (1998) it was discussed how models (37d) and (37e) fail to reproduce the measured values of b_{ij} at the surface of the Sun and that in order to reproduce such data one needs to include vorticity (Z_{ij}) and buoyancy (B_{ij}) in equation (36). In addition, in case the temperature gradient is nonzero only in the z-direction, equation (37e) yields

$$\overline{uv} = 0\,, \quad \overline{u\theta} = \overline{v\theta} = 0\,,$$

which disagrees with laboratory data (Komori et al. 1983; Gerz, Schumann, & Elgobashi 1989; Gerz & Yamazaki 1993). As shown below, the present model is in agreement with the data.

8. VERTICAL DIFFUSIVITIES

In this section we present the analytic solution of equations (36)–(37) for a case of direct astrophysical interest.

The temperature T and large-scale flow U are taken to be

$$\frac{\partial T}{\partial x_i} \to \frac{\partial T}{\partial z}\,\delta_{i_3}\,, \quad U_i = [U(z),\, V(z)\,,\, 0]\,, \tag{38a}$$

and thus shear and vorticity tensors are given by

$$S_{ij} = \tfrac{1}{2}\begin{pmatrix} 0 & 0 & \partial U/\partial z \\ 0 & 0 & \partial V/\partial z \\ \partial U/\partial z & \partial V/\partial z & 0 \end{pmatrix},$$

$$V_{ij} = \tfrac{1}{2}\begin{pmatrix} 0 & 0 & \partial U/\partial z \\ 0 & 0 & \partial V/\partial z \\ -\partial U/\partial z & -\partial V/\partial z & 0 \end{pmatrix}. \tag{38b}$$

In order to homogenize the notation, we introduce the dimensionless variables:

$$\lambda = \frac{\tau_{pv}}{\tau}\,, \quad \lambda_1 = \frac{4}{15}\,\lambda\,, \quad \lambda_2 = \frac{1}{2}(1-\alpha_1)\lambda\,,$$

$$\lambda_3 = \frac{1}{2}(1-\alpha_2)\lambda\,, \quad \lambda_4 = \frac{1}{2}\,\beta_5\,\lambda\,, \quad \lambda_5 = \frac{\tau}{\tau_{p\theta}}\,,$$

$$\lambda_6 = 1 - \frac{3}{4}\,\alpha_3\,, \quad \lambda_7 = 1 - \frac{5}{4}\,\alpha_3\,, \quad \lambda_8 = (1-\gamma_1)\,\frac{\tau_\theta}{\tau}\,. \tag{38c}$$

Equations (36)–(37) can be solved using symbolic algebra. The results are as follows:

Reynolds stresses:

$$\overline{uw} = -\nu_T\,\frac{\partial U}{\partial z}\,, \quad \overline{vw} = -\nu_T\,\frac{\partial V}{\partial z}\,, \tag{39a}$$

$$\overline{uv} = (\lambda_2 + \lambda_3)\tau\nu_T\,\frac{\partial U}{\partial z}\,\frac{\partial V}{\partial z}\,. \tag{39b}$$

Timescales (see eqs. [33a]–[34b]):

$$\frac{1}{2}\,\tau = \frac{K}{\epsilon}\,. \tag{40}$$

Turbulent kinetic energies:

$$\overline{u^2} - \frac{2}{3}\,K = \frac{1}{3}\,\tau\nu_T\left[(\lambda_2 + 3\lambda_3)\left(\frac{\partial U}{\partial z}\right)^2 - 2\lambda_2\left(\frac{\partial V}{\partial z}\right)^2\right] + \frac{2}{3}\,\lambda_4\,\chi_T\,\tau N^2\,, \tag{41a}$$

$$\overline{v^2} - \frac{2}{3}\,K = \frac{1}{3}\,\tau\nu_T\left[(\lambda_2 + 3\lambda_3)\left(\frac{\partial V}{\partial z}\right)^2 - 2\lambda_2\left(\frac{\partial U}{\partial z}\right)^2\right] + \frac{2}{3}\,\lambda_4\,\chi_T\,\tau N^2\,, \tag{41b}$$

$$\overline{w^2} - \tfrac{2}{3}K = \tfrac{1}{3}\tau\nu_T(\lambda_2 - 3\lambda_3)\Sigma^2 - \tfrac{4}{3}\lambda_4\,\chi_T\,\tau N^2\,. \tag{41c}$$

Mean shear:

$$\Sigma^2 \equiv \left(\frac{\partial U}{\partial z}\right)^2 + \left(\frac{\partial V}{\partial z}\right)^2. \tag{42}$$

Mean temperature gradient β (see eq. [2f]):

Heat fluxes:

$$\overline{w\theta} = \chi_T\,\beta \tag{43a}$$

$$\overline{u\theta} = -\lambda_5^{-1}\left[\nu_T + \frac{1}{2}\,(\lambda_6 + \lambda_7)\chi_T\right]\beta\tau\,\frac{\partial U}{\partial z}\,, \tag{43b}$$

$$\overline{v\theta} = -\lambda_5^{-1}\left[\nu_T + \frac{1}{2}\,(\lambda_6 + \lambda_7)\chi_T\right]\beta\tau\,\frac{\partial U}{\partial z}\,, \tag{43c}$$

Turbulent momentum and heat diffusivities:

$$\nu_T = 2S_\nu\,\frac{K^2}{\epsilon}\,, \quad \chi_T = 2S_h\,\frac{K^2}{\epsilon}\,. \tag{44a}$$

Turbulent Prandtl number, σ_T:

$$\sigma_T = \frac{\nu_T}{\chi_T} = \frac{S_\nu}{S_h}\,. \tag{44b}$$

Dimensionless functions, $S_{\nu,h}$:

$$DS_\nu = s_0 + s_1(\tau N)^2 + s_2(\tau\Sigma)^2\,, \tag{45a}$$

$$DS_h = s_4 + s_5(\tau N)^2 + s_6(\tau\Sigma)^2\,, \tag{45b}$$

$$D = d_0 + d_1(\tau N)^2 + d_2(\tau\Sigma)^2 + d_3(\tau N)^4 + d_4(\tau^2 N\Sigma)^2 + d_5(\tau\Sigma)^4\,. \tag{45c}$$

Dimensionless variables, s_k:

$$s_0 = \tfrac{3}{2}\lambda_1\,\lambda_5^2\,, \tag{46a}$$

$$s_1 \equiv -\lambda_4(\lambda_6 + \lambda_7) + 2\lambda_4\,\lambda_5(\lambda_1 - \tfrac{1}{3}\lambda_2 - \lambda_3) + \tfrac{3}{2}\lambda_1\lambda_5\,\lambda_8\,, \tag{46b}$$

$$s_2 \equiv -\tfrac{3}{8}\lambda_1(\lambda_6^2 - \lambda_7^2)\,, \quad s_4 \equiv 2\lambda_5\,, \quad s_5 = 2\lambda_4\,, \tag{46c}$$

$$s_6 = \tfrac{2}{3}\lambda_5(3\lambda_3^2 - \lambda_2^2) - \tfrac{1}{2}\lambda_5\,\lambda_1(3\lambda_3 - \lambda_2) + \tfrac{3}{4}\lambda_1(\lambda_6 - \lambda_7) \tag{46d}$$

Dimensionless variables, d_k:

$$d_0 = 3\lambda_5^2\,, \tag{47a}$$

$$d_1 \equiv \lambda_5(7\lambda_4 + 3\lambda_8)\,, \quad d_2 \equiv \lambda_5^2(3\lambda_3^2 - \lambda_2^2) - \tfrac{3}{4}(\lambda_6^2 - \lambda_7^2)\,, \tag{47b}$$

$$d_3 \equiv \lambda_4(4\lambda_4 + 3\lambda_8)\,, \quad d_5 \equiv \tfrac{1}{4}(\lambda_2^2 - 3\lambda_3^2)(\lambda_6^2 - \lambda_7^2)\,, \tag{47c}$$

$$d_4 \equiv \lambda_4[\lambda_2\,\lambda_6 - 3\lambda_3\,\lambda_7 - \lambda_5(\lambda_2^2 - \lambda_3^2)] + \lambda_5\,\lambda_8(3\lambda_3^2 - \lambda_2^2)\,. \tag{47d}$$

9. FIRST NONLOCAL MODEL

The above relations can be used together with the nonlocal equations for K and ϵ.

Turbulent kinetic energy, K:

$$\frac{DK}{Dt} + D_f(K) = \nu_T\,\Sigma^2 - \chi_T\,N^2 - \epsilon\,, \tag{48}$$

and dissipation, ϵ:

$$\frac{D\epsilon}{Dt} + D_f(\epsilon) = \epsilon K^{-1}(c_1\,\nu_T\,\Sigma^2 - c_3\,\chi_T\,N^2) - c_2\,\epsilon^2 K^{-1}\,. \tag{49}$$

10. SECOND NONLOCAL MODEL

In this model, equation (48) is taken in its algebraic, local form by neglecting the left-hand side altogether. This is physically equivalent to assuming that production = dissipation,

$$\epsilon = \nu_T \Sigma^2 - \chi_T N^2 = \nu_T \Sigma^2 (1 - \mathrm{R}_f) . \tag{50a}$$

Using equations (44) and (40), equation (50a) becomes

$$(\tau\Sigma)^2 S_v - (N\tau)^2 S_h = 2 . \tag{50b}$$

Using equations (45a)–(45c), equation (50b) becomes an equation for the timescale τ (in units of Σ)

$$(\tau\Sigma)^2 \equiv \psi . \tag{51a}$$

The result is

$$A\psi^2 + B\psi + C = 0 , \tag{51b}$$

$$A \equiv (s_5 + 2d_3)\,\mathrm{Ri}^2 - (s_1 - s_6 - 2d_4)\,\mathrm{Ri} - s_2 + 2d_5 ,$$
$$B = (s_4 + 2d_1)\,\mathrm{Ri} - s_0 + 2d_2 , \quad C = 2d_0 \tag{51c}$$

The simplicity of equation (51b) may be deceptive. In fact, when $\mathrm{Pe} > 1$, all the s_k and d_k values are independent of Pe; A, B, C depend only on Ri, and equation (51b) is indeed very simple. However, when $\mathrm{Pe} < 1$, A, B, C depend on Pe, which in turn depends on both (K, ϵ) or (τ, ϵ) (eq. [15]), that is, Pe depends on ψ itself, and equation (51b) is no longer a second-order equation. An expression for Pe is given in the next section.

11. FULLY LOCAL MODEL

One may want to sacrifice even equation (49) by taking its local limit, which we write as

$$\epsilon = \frac{K^{3/2}}{l_\epsilon} . \tag{52}$$

It is indeed a considerable sacrifice for l_ϵ is not easy to model. Before we do so, let us remark that once equation (52) is adopted, K and Pe, given by equation (15), can be expressed as

$$\frac{K}{K_0} = \psi^{-1} , \quad \frac{\mathrm{Pe}}{\mathrm{Pe}_0} = \psi^{-1/2} , \tag{53a}$$

where

$$K_0 = 4 l_\epsilon^2 \Sigma^2 , \quad \mathrm{Pe}_0 = \tfrac{1}{2} c_\epsilon^2 l_\epsilon K_0^{1/2} \chi^{-1} = c_\epsilon^2 l_\epsilon^2 \Sigma \chi^{-1} . \tag{53b}$$

Once a model for l_ϵ is provided, the model becomes fully algebraic.

12. RADIATIVE ENERGY LOSSES

When $\mathrm{Pe} < 1$, we have from equation (35b) that

$$\frac{\tau}{\tau_{p\theta}} > 1 , \quad \frac{\tau}{\tau_\theta} > 1 , \tag{54}$$

and thus from equation (38c) it follows that $\lambda_5 \sim \mathrm{Pe}^{-1} > 1$, $\lambda_8 \sim \mathrm{Pe} < 1$. From equations (46a)–(46d), we then have

$$s_0 \sim \lambda_5^2 \sim \mathrm{Pe}^{-2} , \quad s_{1,4,6} \sim \lambda_5 \sim \mathrm{Pe}^{-1} , \quad s_{2,5} \sim \lambda_5^0 \sim \mathrm{Pe}^0 , \tag{55a}$$

while from equations (47a)–(47d), we have

$$d_{0,2} \sim \lambda_5^2 \sim \mathrm{Pe}^{-2} , \quad d_{1,4} \sim \lambda_5 \sim \mathrm{Pe}^{-1} ,$$
$$d_3 \sim \lambda_5^{-1} \sim \mathrm{Pe} , \quad d_5 \sim \lambda_5^0 \sim \mathrm{Pe}^0 . \tag{55b}$$

13. MODELING THE LENGTH SCALE l_ϵ

We begin by writing l_ϵ as

$$l_\epsilon = c_\epsilon^{-1} \Delta f(N, \Sigma) \tag{56a}$$

and require that $f(0, 0) = 1$. We note that with equation (16), equations (52) and (56a) are consistent with the Kolmogorov energy spectrum $E(k) = \mathrm{Ko}\epsilon^{2/3} k^{-5/3}$. We have called Δ the size of the largest eddy, $k_0 \Delta = \pi$. A plausible model is $\Delta = H_p$ or, what is the same in the case of a polytrope, $\Delta = z$. The distortion of the Kolmogorov spectrum due to stratification and shear is represented by the dimensionless function $f(N, \Sigma)$. Even though a satisfactory theory of the energy spectrum $E(k)$ under shear and stable stratification is still not available, there is some consensus on the overall shape of $E(k)$. Three separate regimes have been identified with the corresponding spectra (Gargett et al. 1981):

$$\begin{aligned} &\mathrm{I}: && E(k) = (\epsilon N)^{1/2} k^{-2} \\ &\mathrm{II}: && E(k) = c N^2 k^{-3} \\ &\mathrm{III}: && E(k) = \mathrm{Ko}\epsilon^{2/3} k^{-5/3} \end{aligned} \tag{56b}$$

Oceanic data (Gargett et al. 1981; Gargett 1989) suggest $c \approx 10$, while atmospheric data suggest $c \approx 100$ (Dewan & Good 1986) so that the interval between regimes I and II may be one or two decades. Furthermore, if one uses the spectra I, II, and III and computes the Froude number, equation (11), one finds that

$$\begin{aligned} &\mathrm{I}: && \mathrm{Fr} < 1 , \quad \mathrm{Ri} > 1 , \\ &\mathrm{II}: && \mathrm{Fr} = 1 , \quad \mathrm{Ri} \sim 1 , \\ &\mathrm{III}: && \mathrm{Fr} > 1 , \quad \mathrm{Ri} < 1 , \end{aligned} \tag{56c}$$

which means that we are mostly interested in regions II and III which coincide at a length scale which is precisely the Ozmidov scale (eq. [13]).

Using empirical arguments rather than a model for the energy spectrum, Deardorff (1980), Hunt et al. (1988), Dubrulle (1993), and Fernando & Hunt (1996) have suggested the relations:

$$f(N, 0) = 0.76 \frac{K^{1/2}}{\Delta N} , \quad f(0, \Sigma) = 2.76 \frac{K^{1/2}}{\Delta\Sigma} . \tag{56d}$$

Several comments are in order. First, since equation (56d) do not satisfy the condition $f(0, 0) = 1$, they have only a limited validity. Second, if in the Ozmidov scale, we substitute equation (52), the resulting length scale is Deardorff's result, the first of equation (56d). Third and most importantly, $f(N, 0)$ implies that

$$\tau \sim \frac{K}{\epsilon} \sim l_\epsilon K^{-1/2} \sim N^{-1} , \quad \tau N = \text{constant} , \tag{57a}$$

which in turn implies that

$$\psi = (\tau\Sigma)^2 \sim \mathrm{Ri}^{-1} . \tag{57b}$$

On the other hand, equation (51b) yields a ψ that increases with Ri; this can be consistent with equation (57b) at only one value of Ri, while Ri must be allowed to assume a range of values. A similar conclusion can be reached using the second of equation (56d).

Several people have tried to construct a model for $E(k, N, \Sigma)$. Lumley's (1964) model can be translated into a function $f(N, \Sigma)$ of the form (Cheng & Canuto 1994):

$$f(N, \Sigma) = [1 - c_1 \, \mathrm{Fr}^{-2} \, (1 - c_2 \sigma_T \mathrm{Ri}^{-1})]^{-3/2} , \quad (57c)$$

where $\sigma_T = \sigma_T(\mathrm{Ri})$. The two constants are given by

$$c_1 \equiv (2\pi^2 3^{1/2})^{-1} \, \mathrm{Ko}^{3/2} , \quad c_2 = 0.4\text{–}0.6 . \quad (57d)$$

The Froude's number is defined as

$$\mathrm{Fr} = \frac{K^{1/2}}{\Delta N} . \quad (57e)$$

As discussed in Cheng & Canuto (1994), Lumley's formulation is valid for small levels of stratification. Weinstock (1978) suggested an improvement of Lumley's model. The corresponding form of $f(N, S)$ was worked out by Cheng & Canuto (1994) to be

$$f(N, \Sigma) = [(1 - A^2 B^2)^{1/2} - AC]^{-3} , \quad (58a)$$

where

$$A \equiv a_1(1 - c_2 \sigma_T \mathrm{Ri}^{-1}) , \quad B^{-1} = a_2 + \mathrm{Fr}^2 ,$$
$$C^{-1} = a_3 + \mathrm{Fr}^2 ,$$
$$a_1 = 8.6810^{-3} \, \mathrm{Ko}^{3/2} , \quad a_2 = 0.025 , \quad a_3 = 0.014 . \quad (58b)$$

Finally, Cheng & Canuto (1994) improved on both Lumley's and Weinstock's model and checked their results against LES data (see their Fig. 5). The function $f(N, \Sigma)$ is given by the following expression:

$$f(N, \Sigma) = [1 + a \, \mathrm{Fr}^{-2} \, (1 + b \, \mathrm{Fr}^{-4/3})^{-1}]^{-c} , \quad (58c)$$

where $\mathrm{R}_f < c_2$:

$$a = 2(3\pi^2)^{-1}(\Omega - 1) , \quad b = 0.12(\Omega - 1 - \tfrac{3}{2}\Omega^{-1})^{4/9} ,$$
$$c = \tfrac{3}{2} , \quad (58d)$$

and $\mathrm{R}_f > c_2$:

$$a = 4(5\pi^2)^{-1}\Omega , \quad b = 0 , \quad c = \tfrac{5}{4}(1 - \Omega^{-1}) ,$$
$$\Omega = 1 + \frac{3^{1/2}}{4} \, \mathrm{Ko}^{3/2} \, (c_2 \sigma_T \mathrm{R}_i^{-1} - 1) . \quad (58e)$$

More recently, Canuto & Cheng (1997) have further improved their model, and their new expression for $f(N, \Sigma)$ is equation (6a) of that paper. In the case in which one considers only shear, $f(0, \Sigma)$ simplifies considerably:

$$2f^{-2/3} = 1 + p \, \mathrm{Sh}^2 + (1 + 2p \, \mathrm{Sh}^2 - \tfrac{4}{3}p^2 \, \mathrm{Sh}^4)^{1/2} , \quad (58f)$$

where

$$\mathrm{Sh} \equiv \Delta\Sigma K^{-1/2} , \quad p = \tfrac{1}{8}(2\pi^2 3^{1/2})^{-1} \, \mathrm{Ko}^{3/2} . \quad (58g)$$

14. TURBULENT PRESSURE

In many instances of astrophysical interest, e.g., in helioseismology (Canuto & Christensen-Dalsgaard 1998) one needs to know the pressure contributed by the turbulent motion itself. This can be seen in the equations for the large-scale flow U, equation (17), where the "effective pressure tensor" is

$$P\delta_{ij} + \rho\overline{u_i u_j} . \quad (59a)$$

When equation (17) becomes the hydrostatic equilibrium equation, the turbulent pressure is

$$p_t = \rho\bar{w}^2 . \quad (59b)$$

Using equation (41c), one obtains

$$p_t(\rho K)^{-1} = \tfrac{2}{3} + \tfrac{1}{3}\psi[(\lambda_2 - 3\lambda_3)S_v - 4\lambda_4 S_h \mathrm{Ri}] . \quad (59c)$$

Not surprisingly, p_t is not a constant fraction of the turbulent kinetic energy.

15. ANALYSIS OF TOWNSEND'S WORK

For a complete treatment of the problem, one needs the dynamic equations for the six variables:

$$K , \quad \bar{\theta}^2 , \quad \overline{uw} , \quad \overline{w\theta} , \quad \epsilon , \quad \epsilon_\theta . \quad (60)$$

Townsend (1958a, 1958b) used only two dynamic equations, for $\bar{\theta}^2$ and K, his equations (3.3) and (3.7). To treat the remaining four, he introduced four unknown functions, k_u, K_θ, L_θ, L_ϵ:

$$k_u = \overline{uw}(\overline{w}^2)^{-1} , \quad k_\theta = \overline{w\theta}(\overline{w}^2\bar{\theta}^2)^{-1/2}$$
$$\epsilon_\theta = \tfrac{1}{3}L_\theta^{-1}K^{1/2}\bar{\theta}^2 , \quad \epsilon = K^{3/2}L^{-1} . \quad (61a)$$

There is a slight difference in the equations for $\bar{\theta}^2$: using equation (29) equations (26a) and (26b) become

$$\beta\overline{w\theta} = \bar{\theta}^2\tau_\theta^{-1} , \quad (61b)$$

where β is defined in equation (2f). Townsend's equation (3.3) reads

$$\beta\overline{w\theta} = \bar{\theta}^2(\tau_\theta^{-1} + \tau_*^{-1}) \equiv \bar{\theta}^2\tau_{\theta*}^{-1} , \quad (61c)$$
$$\tau_\theta^{-1} + \tau_*^{-1} \equiv \tau_{\theta*}^{-1} , \quad (61d)$$

since he included the logarithmic rate of cooling (which we call τ_*^{-1} rather than his "β" to avoid confusion with our β). His length scale L_θ and our τ_θ are related by

$$\tau_\theta^{-1} = \tfrac{1}{3}L_\theta^{-1}(\bar{w}^2)^{1/2} , \quad (61e)$$

while his $L_\epsilon = l_\epsilon$, see his equation (3.7) and our equation (52), which also means that his $\bar{w}^2$ is our K. Thus,

$$\frac{L_\epsilon}{L_\theta} = \frac{3}{2}\frac{\tau}{\tau_\theta} . \quad (61f)$$

Combining Townsend's equations (3.3) and (3.7), one obtains his basic result:

$$\mathrm{R}_f \equiv \mathrm{Ri}\,\sigma_T^{-1} , \quad (62a)$$
$$\sigma_T = \sigma_T(\mathrm{Ri}, \mathrm{Pe}) , \quad (62b)$$
$$\mathrm{R}_f = \tfrac{1}{2}H[1 - (1 - 12A \, \mathrm{Ri} \, H^{-2})^{1/2}] , \quad (62c)$$
$$H = 1 + 3L_\theta\tau_*^{-1}\left(k_u L_\epsilon \frac{\partial U}{\partial z}\right)^{-1} , \quad (62d)$$
$$A = \frac{L_\theta}{L_\epsilon}\left(\frac{k_\theta}{k_u}\right)^2 . \quad (62e)$$

Townsend's relations (62c)–(62e) are correct, but since the model is unable to provide the function σ_T, k_u, k_θ, k_u, the model is unpredictive.

Criterion (3a) was arrived at in the following way: one requires that the square root in (62c) be positive, which implies that

$$\mathrm{Ri} < \frac{H^2}{12}\frac{L_\epsilon}{L_\theta}\left(\frac{k_u}{k_\theta}\right)^2 . \tag{62f}$$

On the other hand, from equations (24) and (29) we have that $\epsilon_\theta \sim \chi\bar{\theta}^2 l^{-2} \sim \tau_\theta^{-1}\bar{\theta}^2$ which gives $\tau_\theta \sim l^2\chi^{-1}$. Equating this with (61e) yields $L_\theta \sim \chi^{-1} w l^2$, and thus

$$\frac{L_\epsilon}{L_\theta} \sim \mathrm{Pe}^{-1} , \tag{62g}$$

which transforms (62f) into

$$\mathrm{Pe\ Ri} < \frac{H^2}{12}\left(\frac{k_u}{k_\theta}\right)^2 . \tag{62h}$$

If one assumes that the right-hand side is constant, one obtains the relation

$$\mathrm{Pe\ Ri} < \mathrm{Ri}_{\mathrm{cr}} , \tag{63a}$$

$$\mathrm{Ri}_{\mathrm{cr}} \equiv \frac{H^2}{12}\left(\frac{k_u}{k_\theta}\right)^2 . \tag{63b}$$

If one further assumed that $\nu_T = 1/3 wl$ and $\mathrm{Pe} = wl/\chi$, one derives the relation

$$\nu_T = \frac{1}{3}\chi\frac{\mathrm{Ri}_{\mathrm{cr}}}{\mathrm{Ri}} , \quad \mathrm{Ri} < \mathrm{Ri}_{\mathrm{cr}} = \frac{1}{4} , \tag{63c}$$

which has been widely employed (e.g., Zahn 1992, 1993, 1994; Maeder 1995, 1996, 1997; Maeder & Meynet 1996).

Let us now use the new turbulence model to compute the ingredients in equation (62c). We employ equations (39a), (43a), (61c) for $\bar{\theta}^2$, and (44). We derive

$$k_u^2 = \psi S_\nu^2 , \quad k_\theta^2 = \tau\tau_{\theta *}^{-1} S_h , \tag{64a}$$

$$\frac{L_\theta}{L_\epsilon} = \frac{2}{3}\frac{\tau_\theta}{\tau} , \quad \left(\frac{k_\theta}{k_u}\right)^2 = \frac{\tau}{\tau_{\theta *}}\sigma_T^{-1}(\psi S_\nu)^{-1} , \tag{64b}$$

$$A = \frac{2}{3}(\psi S_\nu)^{-1}\sigma_T^{-1}\frac{\tau_\theta}{\tau_{\theta *}} , \quad H = 1 + 2\frac{\tau_\theta}{\tau_*}(\psi S_\nu)^{-1} , \tag{64c}$$

where ψ is defined in equation (51a). Equation (62c) then becomes

$$\mathrm{R}_f = \tfrac{1}{2}H\{1 - [1 - 4\mathrm{R}_f \tau_* \tau_{\theta *}^{-1}(H-1)H^{-2}]^{1/2}\} . \tag{65a}$$

This is an equation for R_f which can easily be solved with the result

$$2(\psi S_\nu)^{-1} = 1 - \frac{\mathrm{Ri}}{\phi_T} , \tag{65b}$$

$$\sigma_T = \sigma_T(\mathrm{Ri}, \mathrm{Pe}) . \tag{65c}$$

This is nothing but $P = \epsilon$, equations (50b) and/or (3e) which led to equation (3g) which we have already discussed in § 1.

16. THE Ri → ∞ LIMIT: CONVECTION

Here we show that in the no-shear case, $\mathrm{Ri} \to \infty$, the previous model yields the well-known expressions for the convective flux. When shear vanishes, the only source is buoyancy, which must be positive and thus $N^2 < 0$, $\beta > 0$, $\nabla - \nabla_{\mathrm{ad}} > 0$. From equation (50a) we have

$$\epsilon = \chi_T |N^2| , \tag{66a}$$

or equivalently,

$$\tau^2 |N^2| = 2S_h^{-1} . \tag{66b}$$

Using the definition of S_h, equations (45b) and (45c), we obtain, after some algebra,

$$\tau^2 |N^2| = 3\lambda_5[1 + 4\lambda_4 + 3\lambda_8\tau_\theta/\tau]^{-1} . \tag{66c}$$

From equation (43a), we then have for the convective flux

$$\overline{w\theta} = \chi_T\beta = \beta\chi\Phi , \quad \Phi = \frac{\chi_T}{\chi} = K\tau\chi^{-1}S_h . \tag{66d}$$

Substituting the expression for S_χ and using the definition of the λ values, we finally have

$$\Phi = c_0 S^{1/2}\left(\frac{\tau_{p\theta}}{\tau}\right)^{3/2}\left[1 + c_1\frac{\tau_{pv}}{\tau} + c_2\frac{\tau_\theta}{\tau}\right]^{3/2} , \tag{66e}$$

$$\mathrm{Pe} = c_3(S\Phi)^{1/3} . \tag{66f}$$

Here the timescales values are given by equations (33)–(34), $S = g\alpha\beta\Delta^4\chi^{-2}$, Δ is the size of the largest eddy, and S is related to the convective efficiency Γ (Cox & Giuli 1968) by the expression $2\Gamma + 1 = (1 + 2S/81)^{1/2}$. The coefficients c values are given by $c_0 = (27\pi^{-4})^{1/2}\ \mathrm{Ko}^3$, $c_1 \equiv 2\beta_5$, $c_2 \equiv 3(1-\gamma_1)$, $c_3 = \pi^{2/3}(3\ \mathrm{Ko})^{-1}$. Equation (66e) coincides with equation (42) of Canuto & Dubovikov (1998). It is easy to check that since equations (33)–(34) yield

$$\mathrm{Pe} > 1: \qquad \left(\frac{\tau_{p\theta}}{\tau}, \frac{\tau_\theta}{\tau}\right) \sim \mathrm{Pe}^0 , \tag{66g}$$

$$\mathrm{Pe} < 1: \qquad \left(\frac{\tau_{p\theta}}{\tau}, \frac{\tau_\theta}{\tau}\right) \sim \mathrm{Pe} , \tag{66h}$$

Equations (66e) and (66f) give

$$\mathrm{Pe} > 1: \qquad \Phi \sim \mathrm{Pe} , \quad \Phi \sim S^{1/2} , \tag{66i}$$

$$\mathrm{Pe} < 1: \qquad \Phi \sim \mathrm{Pe}^2 , \quad \Phi \sim S^2 , \tag{66j}$$

which are the well-known limits of the convective flux for large and small convective efficiencies (Cox & Giuli 1968; Canuto & Mazzitelli 1991).

17. THE Ri → 0 LIMIT: PURE SHEAR

In this case, we obtain (to first order) from equations (45a) and (45c)

$$DS_\nu = s_0 , \quad D = d_0 , \quad S_\nu = \tfrac{4}{75} , \tag{67}$$

where we have used equations (38c) and (33a). Thus, the first of equations (44a) becomes

$$\nu_T = C_\nu\frac{K^2}{\epsilon} , \quad C_\nu = 0.1 , \tag{68}$$

which is the well-known formula widely used in shear flows studies, together with the two differential equations (48)–(49) with $N^2 = 0$ to provide K and ϵ (the so called K-ϵ model).

18. FULL MODELS

We have presented three models, two are nonlocal and one is local. The first nonlocal model is characterized by two differential equations for K and ϵ, equations (48) and (49). The nonlocality implies the use of the diffusion terms $D_f(K)$ and $D_f(\epsilon)$ which are given in Appendix A. The model has no mixing length.

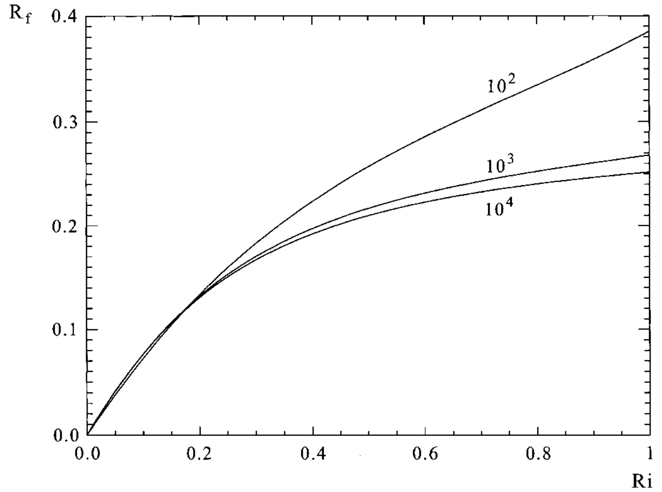

FIG. 5.—Flux Richardson number R_f vs. Ri for different values of Pe_0. As explained in the text, R_f is always less than unity.

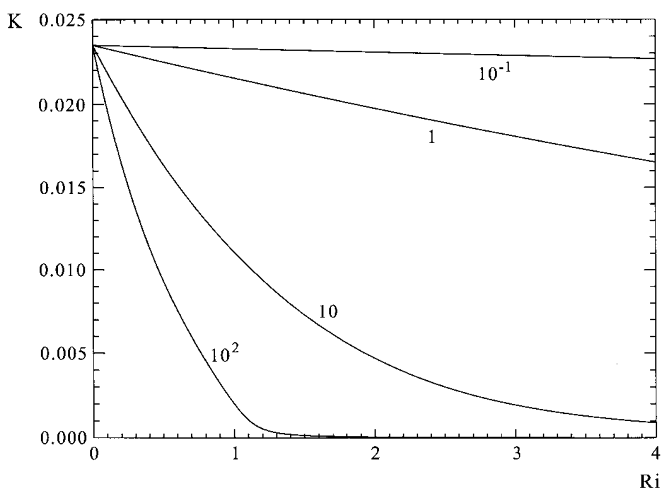

FIG. 6*a*

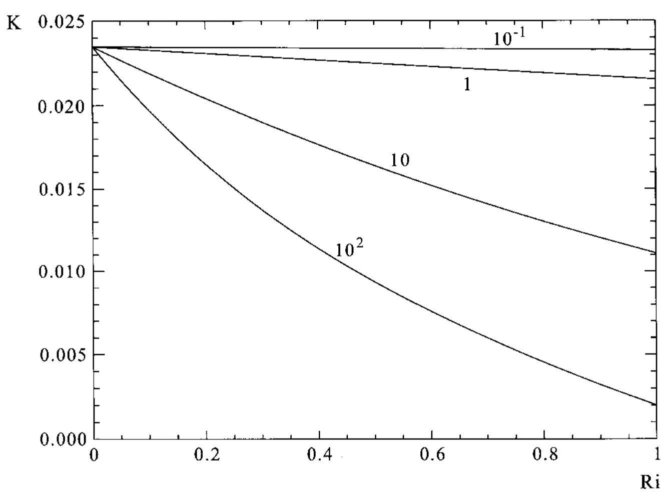

FIG. 6*b*

FIG. 6.—(*a*) Turbulent kinetic energy K, in units of $K_0 = 4l_\epsilon^2 \Sigma^2$ vs. Ri for different values of Pe_0. As expected, the level of turbulence decreases as Ri increases, but when radiative losses are important and stratification becomes weak, the slowdown is considerably reduced. (*b*) Same as in (*a*) for the expanded region up to Ri = 1.

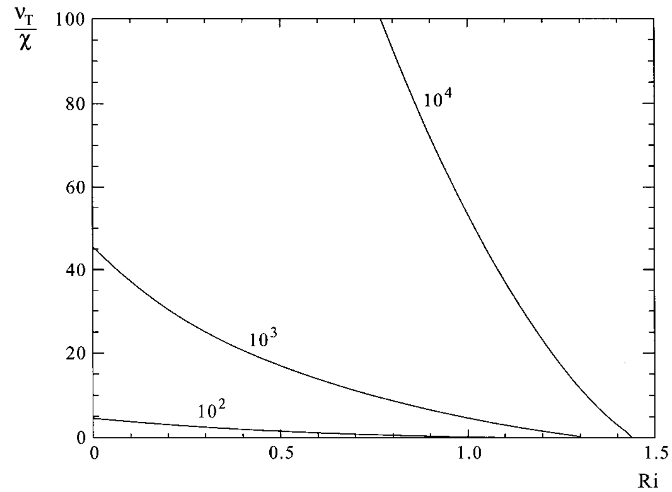

FIG. 7*a*

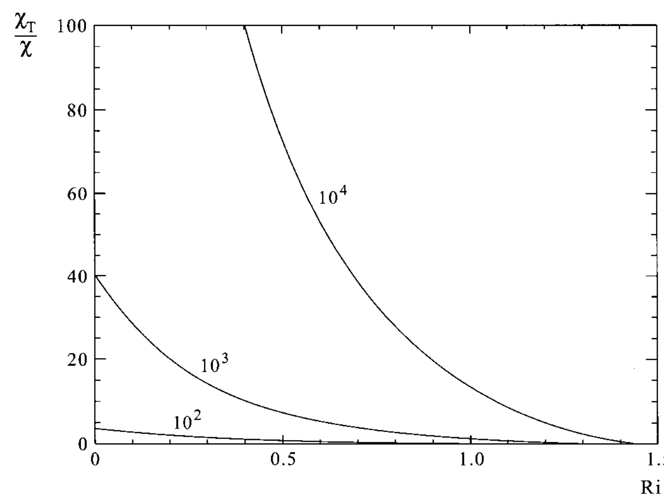

FIG. 7*b*

FIG. 7.—(*a*) Momentum turbulent diffusivity ν_T in units of χ vs. Ri for different Pe_0. As one can see, while the standard model gives no diffusivity for Ri > $\frac{1}{4}$, the present model does yield quite sizable values. (*b*) Same as in (*a*), but for the turbulent heat diffusivity χ_T in units of χ vs. Ri for different Pe_0.

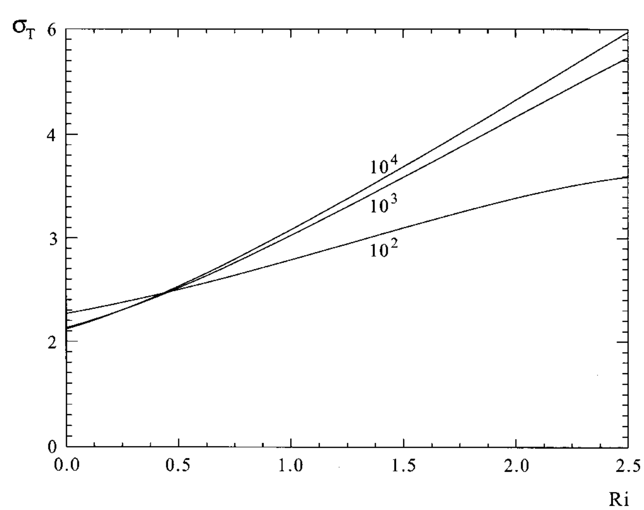

FIG. 8.—Turbulent Prandtl number ν_T/χ_T vs. Ri for different Pe_0

In the local model, both K and ϵ are treated locally, giving rise to equations (51) and (52). There are no differential equations to solve, but the model must specify a mixing length.

19. RESULTS

Before we present some results of the present model, we recall that in the models used thus far ν_T and Pe are taken to be

$$\nu_T = \tfrac{1}{3} wl \,, \quad \mathrm{Pe} = wl\chi^{-1} \,, \tag{69a}$$

and thus

$$3\nu_T \chi^{-1} = \mathrm{Pe} \,. \tag{69b}$$

Second, using the renormalization (3b) valid for small Pe,

$$\mathrm{Pe} = \frac{1}{4\,\mathrm{Ri}} \,. \tag{69c}$$

Equation (69b) becomes

$$\nu_T \chi^{-1} = \frac{1}{12}\frac{1}{\mathrm{Ri}} \,, \quad \mathrm{Ri} < \frac{1}{4} \,. \tag{69d}$$

The present model yields (see eqs. [44a], [15], and [53a]):

$$\nu_T \chi^{-1} = \frac{125}{2\pi^2}\,\mathrm{Pe}_0\, S_\nu\, \psi^{-1/2} \,, \tag{69e}$$

$$\chi_T \chi^{-1} = \frac{125}{2\pi^2}\,\mathrm{Pe}_0\, S_h\, \psi^{-1/2} \,. \tag{69f}$$

First, we solve equation (51b) for ψ in which we use equation (38c) with $\lambda_{5,8}$ depending on Pe via (33b) and (33c) and thus on ψ itself because of equation (53a). The free parameter is Pe_0 (eq. [53b]) which contains basic variables of the problem under consideration. The functions $S_{\nu,h}$ are given by equation (45), and they depend on both Pe and Ri.

In Figure 5 we present the Richardson flux number for different values of the parameter Pe_0 defined in equation (53b) (see also Fig. 4). In Figure 6*a* we present the turbulent kinetic energy K in units of K_0, equations (53a) and (53b) for different Pe_0. The lower the Pe_0, the larger is the value of $\mathrm{Ri}_{\mathrm{cr}}$. In Figure 6*b* we give an expanded version of Figure 6*a* for values up to $\mathrm{Ri} = 1$. In Figure 7*a* we present the momentum turbulent diffusivity ν_T/χ, equation (69e), versus Ri for different Pe_0, while in Figure 7*b* we present the turbulent heat diffusivity χ_T/χ, equation (69f), versus Ri. We must note that in the standard model represented by equation (69d), there is no turbulence and thus no diffusivity, beyond $\mathrm{Ri} = \frac{1}{4}$. In addition, the values resulting from equation (69d) are smaller than those exhibited in Figures 7*a* and 7*b* even for the smallest values of $\mathrm{Pe}_0 = 10^2$. The ratio of the two diffusivities, equation (44b), is shown in Figure 8 (see also Fig. 3).

20. CONCLUSIONS: NATURE OR NURTURE?

Is shear-driven mixing intrinsically weak or did the methodology and approximations used thus far underestimate its real strength? Considering that the constructional incertitudes of all phenomenological models require that one adopts several ad hoc approximations, the doubts are not without justification. The question can be answered only if one employs a turbulence model with a proven record of performance and reliability in flows other than those treated here so that its basic credentials are not in question. The methodologies used thus far are far from being so.

We have documented both physically and mathematically that the approximations made within phenomenological models militate to underestimate the efficiency of shear-driven mixing thus feeding a negative assessment of its real capability. We do not claim to have proved that the new treatment will provide the mixing that stellar data require. We only claim to have employed an internally consistent treatment for all the physical variables, individually and collectively, and to have shown that turbulence is alive and well above the $\mathrm{Ri} > \frac{1}{4}$ limit. The demise of shear-driven mixing may have been announced somewhat prematurely.

Final judgement can, however, only be made after this model is applied to a representative stellar case. Even though the full mode contains 11 differential equations, it seems hardly necessary to begin with such a model. To decide whether the new model provides sufficiently more mixing than the standard model, we think it suffices to first employ the fully algebraic model we have developed.

APPENDIX A

THIRD-ORDER MOMENTS

The equations for the third-order moments are taken from Canuto (1992). In the presence of buoyancy, shear, and rotation, they are

$$\left(\frac{D}{Dt} + \tau_3^{-1}\right)\overline{u_i u_j u_k} = -(\overline{u_i u_j u_l}\, U_{k,l} + \mathrm{perm}) - \left(\overline{u_i u_l}\, \frac{\partial}{l\partial x_l}\, \overline{u_j u_k} + \mathrm{perm}\right) + (1 - c_{11})(\lambda_i \overline{\theta u_j u_k} + \mathrm{perm}) - \frac{2}{3\tau}\,(\delta_{ij}\overline{q^2 u_k} + \mathrm{perm}); \tag{A1}$$

$$\left(\frac{D}{Dt} + \tau_3^{-1}\right)\overline{u_i u_j \theta} = \overline{u_i u_j u_k}\,\beta_k - (\overline{u_i u_k}\,\theta U_{j,k} + \overline{u_j u_k}\theta U_{i,k}) - \left(\overline{u_i u_k}\,\frac{\partial}{\partial x_k}\,\overline{\theta u_j} + \overline{u_j u_k}\,\frac{\partial}{\partial x_k}\,\overline{\theta u_i} + \overline{\theta u_k}\,\frac{\partial}{\partial x_k}\,\overline{u_i u_j}\right) + \frac{2}{3}\,c_{11}\delta_{ij}\lambda_k \overline{\theta^2 u_k} + (1 - c_{11})(\lambda_i \overline{\theta^2 u_j} + \lambda_j \overline{\theta^2 u_j}) \,; \tag{A2}$$

$$\left(\frac{D}{Dt}+\tau_3^{-1}+2\tau_\theta^{-1}\right)\overline{u_i\theta^2}=2\overline{\theta u_i}\,\overline{u_j}\beta_j-\overline{\theta^2}\,\overline{u_j}\,U_{i,j}-2\overline{\theta u_j}\frac{\partial}{\partial x_j}\overline{\theta u_i}+(1-c_{11})\lambda_i\overline{\theta}^3-\overline{u_i u_j}\frac{\partial}{\partial x_j}\overline{\theta}^2\,, \tag{A3}$$

$$\left(\frac{D}{Dt}+\frac{c_{10}}{c_*}\tau_3^{-1}\right)^3=3\beta_j\overline{\theta^2 u_j}-3\overline{\theta u_j}\frac{\partial}{\partial x_j}\overline{\theta}^2+\chi\frac{\partial^2}{\partial x_l^2}\overline{\theta}^3\,. \tag{A4}$$

where $\tau_3 \equiv \tau/2c_*$ and $c_* = 7$, $c_{10} = 4$, $c_{11} = \frac{1}{5}$. The analytical solution of equations (A1)–(A4) in the stationary case can be found in Canuto et al. (1994b). The diffusion of ϵ, $D_f(\epsilon)$ can be found in Canuto & Dubovikov (1998).

APPENDIX B

THE CONSTANTS

For the constants λ, equation (38c), we suggest two sets of values:

$$\lambda_1 = 0.127\,,\quad \lambda_2 = 3.36\times 10^{-3}\,,\quad \lambda_3 = 9.1\times 10^{-2}\,,\quad \lambda_4 = 0.1\,,\quad \lambda_6 = \tfrac{1}{6}\,,\quad \lambda_7 = 0\,,\quad \gamma_1 = \tfrac{1}{3}\,. \tag{B1}$$

$$\lambda_1 = 0.107\,,\quad \lambda_2 = 3.32\times 10^{-3}\,,\quad \lambda_3 = 8.64\times 10^{-2}\,,\quad \lambda_4 = 0.12\,,\quad \lambda_6 = 0.4\,,\quad \lambda_7 = 0\,,\quad \gamma_1 = \tfrac{1}{3}\,. \tag{B2}$$

The values in equations (B2) correspond to the first set of values (eqs. [B1]).

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